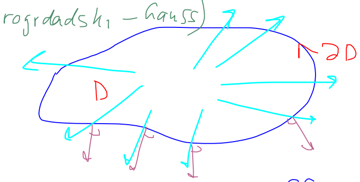
Divergence theorem (Ostrogradski – Gauss)

Green's formula

$\vec F = (P, Q)$

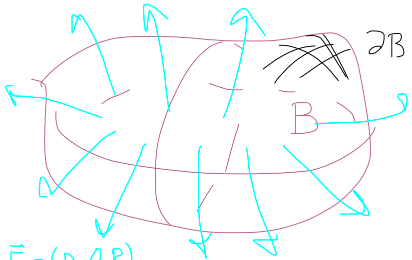


$d\vec r = (dx, dy)$

- $\displaystyle\iint_D (Q_x - P_y)\, dx\wedge dy = \oint_{\partial D} \underbrace{P\,dx + Q\,dy}_{\omega} = \oint_{\partial D} \vec F\cdot d\vec r$

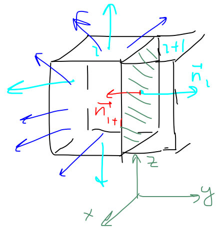
Stoke's formula $\displaystyle\iint_D d\omega = \int_{\partial D} \omega$

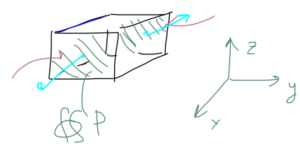
- 2d/1d $\displaystyle\iint_D \operatorname{div} \vec F = \oint_{\partial D} \vec F\cdot\vec n\, ds$



$\vec F = (P, Q, R)$

- 3d/2d $\boxed{\displaystyle\iiint_B \operatorname{div}\vec F\, dx\,dy\,dz = \oiint_{\partial B} \vec F\, d\vec S}$ out, flux





It suffices to prove the theorem for a parallellepiped

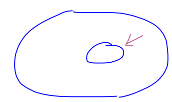
$$\oiint_{\partial B} \vec F\cdot d\vec S = \iint_{\partial B} \vec F\cdot\vec n\, dA = \iint_{\partial B} P\, \vec i\cdot\vec n\, dA + \iint_{\partial B} Q\, \vec j\,\vec n\, dA + \iint_{\partial B} R\, \vec k\,\vec n\, dA$$

$$\parallel \qquad\qquad \parallel \qquad\qquad \parallel$$

$$\iiint_B \frac{\partial P}{\partial x}\, dV + \iiint_B \frac{\partial Q}{\partial y}\, dV + \iiint_B \frac{\partial R}{\partial z}\, dV$$

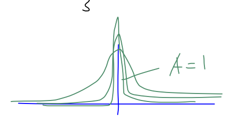
$$= \iiint_B \operatorname{div}\vec F\, dx\,dy\,dz$$

Ex $\quad \vec F = (x, y, z) = x\vec i + y\vec j + z\vec k \qquad \Phi = \oiint_{S^2_R} \vec F\, d\vec S \qquad x^2+y^2+z^2 = R^2$



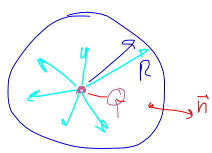
OG $\quad \displaystyle \Phi = \iiint_{B_R} \operatorname{div}\vec F\, dV = \iiint_{B_R} 3\, dV = 3\,\frac{4\pi}{3}R^3 = 4\pi R^3$

$\operatorname{div}\vec F = F_x + F_y + F_z = 1 + 1 + 1$



$\displaystyle\iiint_{\mathbb R^3} \delta^{(3)}\, dV = 1$

Ex



$\vec n = \frac{\vec r}{r}$

$\vec E = \dfrac{Q\,\vec r}{r^3} \qquad \operatorname{div}\vec E \text{ "=" } \begin{cases} 0, & \vec r \neq \vec 0 \\ \infty, & \vec r = 0\end{cases} = 4\pi\, \delta^{(3)}(\vec r)$

claim

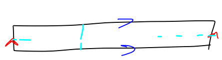
$\displaystyle\iiint \operatorname{div}\vec E = 4\pi Q$

Flux $\quad \displaystyle\Phi = \iint_{S^2} \vec E\cdot\vec n\, dA = \iint_{S^2} \frac{Q}{r^3}\,\vec r\,\frac{\vec r}{r}\, dA$

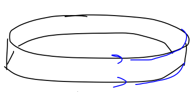
$\displaystyle = \iint_{S^2} \frac{Q}{R^2}\, dA = \frac{Q}{R^2}\iint_{S^2} 1\, dA = \frac{Q}{R^2}\cdot 4\pi R^2 = 4\pi Q$

Orientation

Ex1

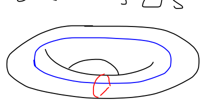


cylinder

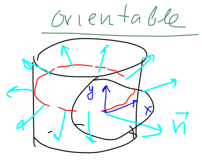


$C\ ,\ \partial C = S^1 \cup S^1$

torus

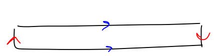


orientable

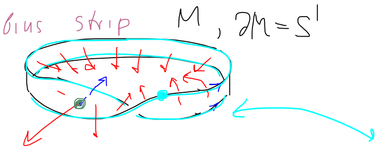


$\vec n = \vec r_x \times \vec r_y$

Ex 2

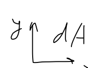


Möbius strip $\quad M,\ \partial M = S^1$



Klein bottle

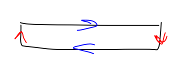
not orientable



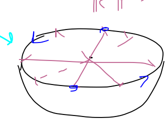
$dA = dx\wedge dy = -dy\wedge dx$

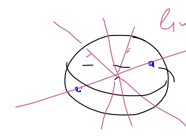
Integrals $\displaystyle 0 = \iint_M f\, dx\wedge dy = -\iint_M f\, dy\wedge dx$

Ex 3.



Real projective plane $\mathbb{RP}^2$





line in $\mathbb R^3$ passing through = pt in $\mathbb{RP}^2$