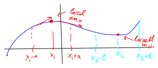
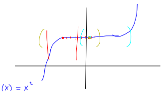
Week 5 Midterm on Thursday. 25 hrs

(14.7) Minimum and Maximum values

1-variable





If $f(x_1) \geq f(x)$ $\forall x \in (x_1-a, x_1+a)$

x_1 - local max $\Rightarrow$ $f'(x_1)=0$, $f''(x_1) \leq 0$

If $f(x_2) \leq f(x)$ $\forall x \in (x_2-b, x_2+b)$

x_2 - local min $\Rightarrow$ $f'(x_2)=0$, $f''(x_2) \geq 0$

$f(x) = x^2$

$x=0$ - global min

$y=x^3$

$y' = 3x^2|_{x=0} = 0$

Def $f(x,y)$ has a local maximum (minimum) at (a,b) if for all (x,y) from a small open disk centered at (a,b) $\leftarrow$ $\forall x$ $\sqrt{(x-a)^2+(y-b)^2} \leq \delta$

$f(x,y) \leq f(a,b)$ $\quad (f(x,y) \geq f(a,b))$

local maximum

local minimum

max, min $\Rightarrow$ extrema

Theorem If f has a local extremum at $(x,y)=(a,b)$ and its f_x, f_y exist at (a,b) then $f'_x(a,b) = f'_y(a,b) = 0$

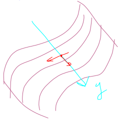
Proof Let $g(x) = f(x,b)$, so $g(x)$ has a local extremum at $x=a$

Fermat's theorem $g'(a)=0$ $\quad g'(a) = f'_x(x,b) = 0$

$h(y) = f(a,y)$, $h'(b)=0 \Rightarrow f'_y(a,b)=0$

The opposite is not always true!

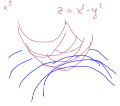
$f(x,y) = x^3$



$f(x_1 \; x_n) \sim x_1^2 - x_2^2 + x_3^2 - x_1^2 -$

$z = x^2 - y^2$

$(x,y)=(0,0)$
saddle point



Theorem Suppose second partial derivatives are continuous at a small disc centered at (a,b), $f_x(a,b)=0$, $f_y(a,b)=0$

Define $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$

a) If $D>0$ and $f_{xx}(a,b)>0$ $\Rightarrow$ $f(a,b)$ is a local minimum

b) If $D>0$ and $f_{xx}(a,b)<0$ $\Rightarrow$ $f(a,b)$ is a local maximum

c) If $D<0$ then $f(a,b)$ is not a maximum, nor minimum

$\Delta x = x - x_0$

$$\Delta f = \cancel{f'(x_0)}^{0} \Delta x + \underbrace{\frac{1}{2!} f''(x_0)(\Delta x)^2}_{} + \ldots$$

$\Delta x = x - x_0$
$\Delta y = y - y_0$

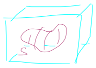
$$\Delta f(x,y) = f'_x(x_0,y_0)dx + f'_y(x_0,y_0)dy + \overbrace{\frac{1}{2!}(dx \;\; dy)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}}^{d^2 f} + \frac{1}{3!}\sum_{a,b,c=1}^{2}(dx)^a(dy)^b(dz)^c f'''_{abc} + \ldots$$

$f'''_{xxx}(dx)^3 + \ldots$

Absolute minima and maxima

~~[||||||]~~ $0 \; \{\frac{1}{n}\} \to 0 \; 1$

compact = closed & bounded (containing all limiting pts)



$\exists B>0$ s.t. $|\vec{x}| \leq B$ $\forall \vec{x} \in S$

Theorem Let $f(x_1 \; x_n)$ be continuous on $K \subset D$ - domain (compact subset) then f attains its minimum and maximum on K

1D $S=(0,1)$ $\quad f = \frac{1}{x}$ - no maximum, nor minimum

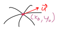
$\sin x$ on $[0,1]$

Def We have a Global maximum (minimum) at $(a,b) \in D$ if $f(x,y) \leq f(a,b)$ $(f(x,y) \geq f(a,b))$

$\forall (x,y) \in D$

Algorithm
1) Find local extrema $\quad f'_x = f'_y = 0$, $D \overset{?}{>}{<} 0$
2) Check the boundary

Claim If $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} > 0$, $f_{xx}>0 \Rightarrow$ local minimum



Proof $\partial_{\vec{u}} f = f_x h - f_y k$

$\vec{u} = (h,k)$
$h^2+k^2=1$

$$\partial^2_{\vec{u}} f = \partial_{\vec{u}}(f_x h + f_y k) = \frac{\partial}{\partial x}(f_x h + f_y k)h + \frac{\partial}{\partial y}(f_x h + f_y k)k$$

$$= f_{xx}h^2 + f_{xy}hk + f_{yx}hk + f_{yy}k^2$$

$$= f_{xy}h^2 + 2f_{xy}hk + f_{yy}k^2$$

$$= f_{xx}\left(h^2 + 2\frac{f_{xy}}{f_{xx}}hk + \left(\frac{f_{xy}}{f_{xx}}\right)^2 k^2\right) - \frac{f_{xy}^2}{f_{xx}}k^2 + f_{yy}k^2$$

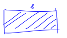
$$\partial^2_{\vec{u}} f = \underbrace{f_{xy}\left(h + \frac{f_{xy}}{f_{xx}}k\right)^2}_{\vee 0} + \frac{h^2}{\underset{\vee 0}{f_{xx}}}\underbrace{\left(f_{xx}f_{yy} - f_{xy}^2\right)}_{D>0} > 0$$

$\Rightarrow$ local minimum

$f_{xx}(a,b)>0$

MT Weeks 1-4 all up to 14.7 (not including)

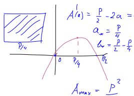
Lagrange multipliers



Maximum area for fixed perimeter $\quad P = 2a+2b$

$b = \frac{P}{2} - a$

$A_{(a)} = ab = a(\frac{P}{2} - a) = a\frac{P}{2} - a^2$

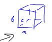


$A'(a) = \frac{P}{2} - 2a = 0$

$a_* = \frac{P}{4}$

$b_* = \frac{P}{2} - \frac{P}{4} = \frac{P}{2}$

$A_{max} = \frac{P^2}{16}$



$A = 2ab + 2ac + 2bc$

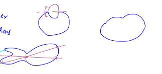
$V_{max} = ?$

$V = abc$

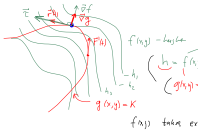
L Length fixed

Q what shape maximizes the area?

1) Shape should be convex
2) Symmetric wrt to rotations center-symmetric



$A_{max} = \frac{\pi L^2}{4}$



Find extremum of $f(x,y)$ provided that $g(x,y)=K$

$\left(\underset{\text{level curves}}{h = f(x,y)} \right.$ gradient $\vec{\nabla} f$

$\left. (g(x,y)=K) \right)$

Find maximal h s.t. these two curves intersect

$\vec{r}'(t) \parallel \vec{\tau} \Rightarrow \underset{\text{Lagrange multiplier}}{\lambda}\vec{\nabla} g = \vec{\nabla} f$

$f(x,y)$ takes extremum value at (x_0,y_0) if $\exists \lambda \in \mathbb{R}$ s.t. $\boxed{\begin{array}{l}\vec{\nabla} f(x_0,y_0) = \lambda \vec{\nabla} g(x_0,y_0) \\ \text{and} \quad g(x_0,y_0) = K\end{array}}$

$\vec{\nabla} f = (f_x, f_y)$
$\vec{\nabla} g = (g_x, g_y)$

$$\left[\begin{array}{l} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = K \end{array}\right. \Big\} \Leftarrow \vec{\nabla} f = \lambda \vec{\nabla} g$$

New function $S(x,y,\lambda) = f(x,y) - \lambda(g(x,y) - K)$

Extremize S

$S_x = 0 \Rightarrow f_x - \lambda g_x = 0$
$S_y = 0 \Rightarrow f_y - \lambda g_y = 0$
$S_\lambda = 0 \Rightarrow g - K = 0$

(Lagrange function)

Ex

$\mathcal{L} = m\frac{(x'(t))^2}{2} - V(x(t), x'(t))$

$S = \int_\gamma \mathcal{L}(x, x', t)dt$
action

$mx'' = -\vec{\nabla} V$
$\shortparallel$
F

Ex. Volume of parallelepiped $\quad a = \frac{A}{2}$

$$S(x,y,z,\lambda) = \underbrace{xyz}_{f(x,y,z)} - \lambda \underbrace{(xy + xz + yz - a)}_{g(x,y,z)=a}$$

$S_x = yz - \lambda y - \lambda z = 0$
$S_y = xz - \lambda x - \lambda z = 0$
$S_z = xy - \lambda x - \lambda y = 0$
$S_\lambda = xy + xz + yz - a = 0$

$\vec{\nabla} f = \lambda \vec{\nabla} g$

$g = K$

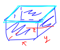
$\cdot x$
$\cdot y$
$\cdot z$

$xyz = \lambda(xy + xz)$
$xyz = \lambda(xy + yz)$
$xyz = \lambda(xz + yz)$

$\cancel{xy} + xz = \cancel{xy} + yz \Rightarrow x = y$
$xy + \cancel{yz} = xz + \cancel{yz} \Rightarrow y = z$
$\Rightarrow x = y = z$

$xy + xz + yz = a$
$3x^2 = a$ $\quad x = +\sqrt{\frac{a}{3}}$

$V = \left(\frac{a}{3}\right)^{3/2} = \left(\frac{A}{6}\right)^{3/2}$



$A = 2xy + 2yz + xy$

$x = y = 2z$

Ex $f(x,y) = ax^2 + by^2$ Find extreme values of f on $x^2+y^2=1$

$a \neq b$

$S(x,y,\lambda) = ax^2 + by^2 - \lambda(x^2+y^2-1)$

$S_x = 2ax - 2\lambda x = 0$
$S_y = 2by - 2\lambda y = 0$
$S_\lambda = x^2+y^2-1 = 0$

$$\begin{cases} ax = \lambda x \Rightarrow \left[\begin{array}{l} x=0 \\ \lambda=a \end{array}\right. \\ by = \lambda y \rightarrow b \cdot 1 = \lambda \cdot 1 \\ x^2+y^2 = 1 \end{cases}$$

$\begin{cases} x=0 \\ y=\pm 1 \\ \lambda=b \end{cases}$

$\lambda = a$

$\left[\begin{array}{l} y=0 \\ \lambda=b \end{array}\right.$

$\begin{cases} x=\pm 1 \\ y=0 \\ \lambda=a \end{cases}$

$f(0, \pm 1) = b$ - max $\quad b > a$

$f(\pm 1, 0) = a$ - min

Ex Sphere $x^2+y^2+z^2=4$, $P(3,1,-1)$ point A on S^2 s.t. distance $|AP|$ is the shortest

$A = (x,y,z)$ $\quad d^2(x,y,z) = (x-3)^2 + (y-1)^2 + (z+1)^2$

$S(x,y,z) = d^2(x,y,z) - \lambda(x^2+y^2+z^2-4)$

$S_x = 2(x-3) - 2\lambda x = 0$
$S_y = 2(y-1) - 2\lambda y = 0$
$S_z = 2(z+1) - 2\lambda z = 0$
$S_\lambda = x^2+y^2+z^2-4 = 0$

$x(1-\lambda) = 3$
$y(1-\lambda) = 1$
$z(1-\lambda) = -1$

$x = \frac{3}{1-\lambda}$
$y = \frac{1}{1-\lambda}$
$z = -\frac{1}{1-\lambda}$

$9 + 1 + 1 = 4(1-\lambda)^2$

$(1-\lambda)^2 = \frac{11}{4}$ $\quad \lambda = 1 \pm \frac{\sqrt{11}}{2}$

$x_\pm = \mp\frac{6}{\sqrt{11}}$
$y_\pm = \mp\frac{2}{\sqrt{11}}$
$z_\pm = \pm\frac{2}{\sqrt{11}}$

$A' = \left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$

$A = \left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$

Extremization problems with several constraints

$f(x_1 \; x_n)$, $g(x_1 \; x_n) = 0$

1 constraint $\quad \boxed{S(x_1 \; x_n, \lambda) = f(x_1 \; x_n) - \lambda g(x_1 \; x_n)}$

$$\vec{\nabla} S = 0 \Rightarrow \begin{cases} S_{x_1} = 0 \\ S_{x_2} = 0 \\ \\ S_{x_n} = 0 \\ S_\lambda = 0 \end{cases}$$

m constraints $\quad f(x_1 \; x_n)$, $\quad g_1(x_1 \; x_n) = 0$
$g_2(x_1, \; x_n) = 0$

$g_m(x_1 \; x_n) = 0$

$$S(x_1 \; x_n, \lambda_1 \ldots \lambda_m) = f(x_1 \; x_n) - \lambda_1 g_1(x_1, x_n) - \ldots - \lambda_m g_m(x_1 \ldots x_n)$$

$\vec{\nabla} S = 0 \Rightarrow$

$S_{x_1} = 0$
$S_{x_n} = 0$ $\Big\}$ $\quad f_{x_1} = \lambda_1(g_1)_{x_1} + \lambda_2(g_2)_{x_1} + \ldots + \lambda_m(g_m)_{x_1}$
$\quad f_{x_n} = \lambda_1(g_1)_{x_n} + \lambda_2(g_2)_{x_n} + \ldots + \lambda_m(g_m)_{x_n}$

$S_{\lambda_1} = 0$
$S_{\lambda_m} = 0$ $\Big\}$ $\quad g_1 = 0$
$\quad g_m = 0$

$n+m$ eqns

$n+m$ unknowns

$\vec{\nabla} f = \lambda_1 \vec{\nabla} g_1 + \ldots + \lambda_m \vec{\nabla} g_m$

$\boxed{\vec{\nabla} f = \sum_{i=1}^{m} \lambda_i \vec{\nabla} g_i}$

Ex $f = x + 2y + 3z$ $\quad \begin{cases} g_1 = x - y + z - 1 = 0 \\ g_2 = x^2+y^2-1 = 0 \end{cases}$

$$S(x,y,z,\lambda_1,\lambda_2) = x + 2y + 3z - \lambda_1(x-y+z-1) - \lambda_2(x^2+y^2-1)$$

$S_x = 1 - \lambda_1 - 2\lambda_2 x = 0$
$S_y = 2 + \lambda_1 - 2\lambda_2 y = 0$
$S_z = 3 - \lambda_1 = 0 \rightarrow \lambda_1 = 3$
$S_{\lambda_1} = x - y + z - 1 = 0 \rightarrow z = 1 - x + y = 1 \pm \frac{7}{\sqrt{29}}$
$S_{\lambda_2} = x^2+y^2-1 = 0 \rightarrow 1 + \frac{25}{4} = \lambda_2^2$

$\begin{cases} -2 - 2\lambda_2 x = 0 \\ 5 - 2\lambda_2 y = 0 \end{cases}$

$\begin{cases} \lambda_2 x = -1 \\ \lambda_2 y = 5/2 \end{cases}$

$x = -\frac{1}{\lambda_2}$

$y = \frac{5}{2\lambda_2}$

$\lambda_2^2 = \frac{29}{4}$ $\quad \lambda_2 = \pm\frac{\sqrt{29}}{2}$

$x_\pm = \mp\frac{2}{\sqrt{29}}$
$y_\pm = \pm\frac{5}{\sqrt{29}}$
$z_\pm = 1 \pm \frac{7}{\sqrt{29}}$

$f_\pm = f(x_\pm, y_\pm, z_\pm) = \ldots = 3 \pm \sqrt{29}$

f_+ - max

f_- - min