Name (Last, First): ____________________

Student ID: ____________________

Circle your GSI and section:

Scerbo	8am	200 Wheeler	Forman	2pm	3109 Etcheverry
Scerbo	9am	3109 Etcheverry	Forman	4pm	3105 Etcheverry
McIvor	12pm	3107 Etcheverry	Melvin	5pm	24 Wheeler
McIvor	11am	3102 Etcheverry	Melvin	4pm	151 Barrows
Mannisto	12pm	3 Evans	Mannisto	11am	3113 Etcheverry
Wayman	1pm	179 Stanley	McIvor	2pm	179 Stanley
Wayman	2pm	81 Evans			

If none of the above, please explain: ____________________

This exam consists of 10 problems, each worth 10 points, of which you must complete 8. **Choose two problems not to be graded by crossing them out in the box below.** You must justify every one of your answers unless otherwise directed.

Problem	Maximum Score	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total Possible	80	

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1. Let V be a nonzero finite-dimensional real vector space. Suppose $T : V \to V$ is a linear transformation.

Decide if the following assertions are ALWAYS TRUE or SOMETIMES FALSE. You need not justify your answer.

i. There exists an eigenvalue of T.
F

ii. There exists a basis of V such that T is upper-triangular.
F

iii. $\dim V = \dim null(T) + \dim range(T)$
T

iv. If v and w are colinear, then Tv and Tw are colinear.
T

v. If v and w are linearly independent, then Tv and Tw are linearly independent.
F

vi. If T is invertible and λ is an eigenvalue of T, then λ^{-1} is an eigenvalue of T^{-1}.
T

vii. If T is invertible and v is an eigenvector of T, then v is an eigenvector of T^{-1}.
T

viii. If $T^2 = 1$, then T has an eigenvalue.
T

ix. If $T^3 = T^2$, then T has an eigenvalue.
T

x. If $T^3 = T^2$, then $null(T) \neq \{0\}$.
F

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2. Let V be an inner product space and $v_1, \ldots, v_n$ a list of vectors in V.

(a) State what it means for $v_1, \ldots, v_n$ to be linearly independent. State what it means for $v_1, \ldots, v_n$ to be orthonormal.

$v_1, \ldots, v_n$ is linearly independent means whenever $a_1 v_1 + \cdots a_n v_n = 0$ for scalars $a_1, \ldots, a_n$, we have that $a_1 = \cdots = a_n = 0$.

$v_1, \ldots, v_n$ orthonormal means $\langle v_i, v_j \rangle$ is equal to 0 if $i \neq j$ and is equal to 1 when $i = j$.

(b) Prove that if $v_1, \ldots, v_n$ is orthonormal, then $v_1, \ldots, v_n$ is linearly independent.

Suppose $a_1 v_1 + \cdots a_n v_n = 0$. Then for all $i = 1, \ldots, n$, we have $0 = \langle a_1 v_1 + \cdots a_n v_n, v_i \rangle = a_1 \langle v_1, v_i \rangle + \cdots a_n \langle v_n, v_i \rangle = a_i \langle v_i, v_i \rangle = a_i$. Thus we have that $a_1 = \cdots = a_n = 0$.

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3. Let $A \in M_{n\times n}(\mathbb{C})$ be a complex matrix. Consider the subspace $W \subset M_{n\times n}(\mathbb{C})$ given by

$$W = span\{I, A, A^2, A^3, \ldots, A^k, \ldots\}$$

Prove that

$$\dim W \leq n.$$

By the Cayley-Hamilton Theorem, we have $\chi_A(A) = 0$ where $\chi_A(z)$ is the characteristic polynomial. Recall that $\chi_A(z)$ is monic of degree n, and thus A^n is in the span of $I, A, \ldots, A^{n-1}$. For any $k \geq 1$, we similarly have that A^{n+k} is in the span of $A^k, A^{k+1}, \ldots, A^{n+k}$. Thus by induction, we have that A^{n+k} is in the span of $I, A, \ldots, A^{n-1}$.

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4. Consider $\mathbb{C}^3$ with the standard Euclidean inner product. Determine whether each of the following operators $T : \mathbb{C}^3 \to \mathbb{C}^3$ is self-adjoint, normal, or neither. You need not justify your answer.

a. T has eigenvectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ with respective eigenvalues $0, 1 + i, 1 - i$.
Normal but not self-adjoint.

b. T has eigenvectors $(1, i, 0)$, $(1, -i, 0)$, $(0, 0, 1)$ with respective eigenvalues $1, -1, 0$.
Self-adjoint.

c. T has eigenvectors $(1, 0, 0)$, $(0, i, -i)$, $(1, 1, 1)$ with respective eigenvalues $1, -1, 1$.
Self-adjoint.

d. $\dim null(T^2) = 3$, $\dim range(T) = 1$.
Neither.

e. $\dim null(T - i) = 2$, $\dim null(T) = 1$ with $null(T - i) \perp null(T)$.
Normal but not self-adjoint.

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5. Find a basis for $\mathbb{C}^3$ that puts the operator given by the matrix

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

into Jordan canonical form. What is the Jordan canonical form?

Take $v_1 = Av_2 = (0,0,1)$, $v_2 = Av_3 = (0,1,1)$, $v_3 = (1,0,0)$.

Jordan form:

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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6. Consider $\mathbb{R}^2$ with the inner product

$$\langle (x_1, x_2), (y_1, y_2) \rangle = 2x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2$$

a. Find an orthonormal basis for $\mathbb{R}^2$ with respect to the above inner product.
Take $e_1 = (0, 1)$, $e_2 = (1, -1)$.

b. Find the vector $v = (a, b)$ closest to $(1, 0)$ satisfying $a + b = 0$.
$v = (1, -1) = \langle (1, 0), e_2 \rangle e_2$.

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7. Find the Jordan form of an operator $T : \mathbb{C}^5 \to \mathbb{C}^5$ given the following information:

$\dim null(T^2) = 2 \qquad \dim null(T^3) = 3 \qquad \dim null((T-1)^2) = 2 \qquad \dim range(T-1) = 4$

Be sure to justify your answer.

Jordan form:

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $\dim null(T^3) = 3$, we have $\dim \tilde{E}_0 \geq 3$. Since $\dim null((T-1)^2) = 2$, we have $\dim \tilde{E}_1 \geq 2$. Thus since $\dim \mathbb{C}^5 = 5$, we must have $\dim \tilde{E}_0 = 3$ and $\dim \tilde{E}_1 = 2$.

Next, since $\dim null(T^2) = 2$, we must have the Jordan block

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since $\dim range(T-1) = 4$, we can not have $\dim null(T-1) = 2$, and so we must have the Jordan block

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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8. Consider the following matrices:

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad T_5 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad T_6 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Which of the matrices has minimal polynomial $m(z) = z^3 + z$? Be sure to justify your answer.

T_2, T_3, T_6. They each satisfy $m(z) = z^3 + z = z(z - i)(z + i) = 0$ and have each eigenvalue so no factor can be removed.

$i, -i$ are not eigenvalues of T_1, T_4.

T_5 has a Jordan block

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which means its minimal polynomial must contain z^2 as a factor.

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9. Consider the matrix

$$T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Calculate T^{100} applied to the vector $(3,2)$.

T has eigenvectors $(1,1)$, $(1,-1)$ with respective eigenvalues $0,2$.
$(3,2) = \frac{5}{2}(1,1) + \frac{1}{2}(1,-1)$.
Thus $T^{100}(3,2) = T^{100}(\frac{5}{2}(1,1) + \frac{1}{2}(1,-1)) = T^{100}(\frac{1}{2}(1,-1)) = 2^{99}(1,-1)$.

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10. Let V be a complex vector space of dimension n. Suppose $T : V \to V$ satisfies $T^n = 0$ but $T^{n-1} \neq 0$. Show that there is a vector $v \in V$ such that the list $v, Tv, T^2v, \ldots, T^{n-1}v$ is a basis.

Since $T^{n-1} \neq 0$, there exists a vector $v \in V$ such that $T^{n-1}v \neq 0$.

Suppose $a_1 v + a_2 Tv + \cdots + a_n T^{n-1}v = 0$. Apply T^{n-1} to obtain $a_1 T^{n-1}v = 0$. Thus $a_1 = 0$ and so $a_2 Tv + \cdots + a_n T^{n-1}v = 0$.

Apply Apply T^{n-2} to obtain $a_2 T^{n-1}v = 0$. Thus $a_2 = 0$ and so $a_3 T^2 v + \cdots + a_n T^{n-1}v = 0$.

Keep repeating to conclude $a_1 = \cdots = a_n = 0$.

Thus $v, Tv, T^2v, \ldots, T^{n-1}v$ is linearly independent. Since it has size $n = \dim V$, it also must span and hence be a basis.