Homework 12 Solutions

1. Clearly $T^2 = 0$, so every $(w, z) \in \mathbb{C}^2$ is a generalized eigenvector of T.

2. Elementary calculations show that the eigenvalues of T are $i, -i$ and that corresponding eigenvectors are $(1, -i), (1, i)$, respectively. Since these span $\mathbb{C}^2$, the only generalized eigenvectors are $\mathrm{Span}((1, -i)) \cup \mathrm{Span}((1, i))$.

3. Suppose we have scalars $c_0, c_1, ..., c_{m-1}$ such that $c_0 v + c_1 T v + ... + c_{m-1} T^{m-1} v = 0$. Apply T^{m-1} to both sides to obtain that $c_0 T^{m-1} v = 0$. Since $T^{m-1} v \neq 0$, we conclude that $c_0 = 0$.

   Suppose now we have shown that $c_0 = c_1 = ... = c_{j-1} = 0$ for some j. Then we have $c_j T^j v + c_{j+1} T^{j+1} v + ... + c_{m-1} T^{m-1} v = 0$ Apply T^{m-j-1} to both sides to obtain $c_j T^{m-1} v = 0$, whence $c_j = 0$.

4. Note that $T^3 = 0$, but $T^2 \neq 0$. Suppose we had $S \in \mathcal{L}(V)$ such that $S^2 = T$. Then we would have $S^6 = T^3 = 0$, so S is nilpotent. Thus, $S^{\dim \mathbb{C}^3} = S^3 = 0$, so also $T^2 = S^4 = 0$, a contradiction.

5. Choose k such that $(ST)^k = 0$. Then $(TS)^{k+1} = T(ST)^k S = 0$.

6. Choose k such that $N^k = 0$, let λ be an eigenvalue of N, and let v be a corresponding nonzero eigenvector. Then $\lambda^k v = N^k v = 0$, whence $\lambda = 0$.

7. Trivial consequence of 6 and the Spectral Theorem

10. Any nonzero nilpotent operator $N \in \mathcal{L}(V)$ satisfies $\mathrm{null}\, N \cap \mathrm{range}\, N \neq \{0\}$, so it cannot happen that $V = \mathrm{null}\, N \oplus \mathrm{range}\, N$.

11. First, note that Rank-Nullity guarantees that $\dim V = \dim \mathrm{null}\, T^n + \dim \mathrm{range}\, T^n$, so we need only show that $\mathrm{null}\, T^n \cap \mathrm{range}\, T^n = \{0\}$.

Let v be some vector in $\operatorname{null} T^n \cap \operatorname{range} T^n$. Since $v \in \operatorname{range} T^n$, we may choose some w such that $v = T^n w$. Since $v \in \operatorname{null} T^n$, we have that $T^{2n} w = T^n v = 0$, so $w \in \operatorname{null} T^{2n}$. But by Proposition 8.6, $\operatorname{null} T^{2n} = \operatorname{null} T^n$, so $v = T^n w = 0$.

12. Put N into upper-triangular form. Then N has all 0's on the diagonal. Clearly any such matrix is nilpotent; hence, so is N.

    For a counterexample on a real vector space, consider the operator on $\mathbb{R}^3$ whose matrix with respect to the standard basis is

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

13. Arguing as in the proof of Proposition 8.6, we see that $\dim T^{n-1} \geq n-1$. Since generalized eigenvectors corresponding to distinct eigenvalues are linearly independent, there is room for at most one more eigenvalue.