Solutions to Final Exam.
1. i. T ii. T iii. T iv. T v. F vi. T vii. T viii. F ix T x. T

2. (a) $v_1, \ldots, v_k$ linearly independent means if for any scalars $a_1, \ldots, a_k$, we have $a_1v_1 + \cdots + a_kv_k = 0$, then we have $a_1 = \cdots = a_k = 0$. The span of $v_1, \ldots, v_k$ is the subset of V comprising vectors v that can be written in the form $v = a_1v_1 + \cdots + a_kv_k$ for some scalars $a_1, \ldots, a_k$. (b) Since $v \in Span(v_1, \ldots, v_k)$, there are scalars $c_1, \ldots, c_k$ in F such that

$$v = c_1v_1 + \cdots c_kv_k.$$

This equation is a nontrivial linear dependence amongst the vectors $(v_1, \ldots, v_k, v)$ (nontrivial since at least the coefficient of v is nonzero), thus this list is dependent.

3. Since T is normal, the complex spectral theorem applies, and we know there is an orthonormal basis $(u_1, \ldots, u_n)$ for V consisting of eigenvectors for T. Pick any $v \in V$, and write it as $v = c_1u_1 + \cdots c_nu_n$. Since this is an orthonormal basis we have $\|v\|^2 = |c_1|^2 + \cdots + |c_n|^2$. Next, let the eigenvalues of u_i be λ_i, so we have

$$Tv = c_1\lambda_1u_1 + \cdots c_n\lambda_nu_n,$$

hence using the fact that this is an orthonormal basis again, and that $|\lambda_i| \leq 1$,

$$\|Tv\|^2 = \|c_1\lambda_1u_1 + \cdots c_n\lambda_nu_n\|^2 = |c_1|^2|\lambda_1|^2 + \cdots + |c_n|^2 \leq |c_1|^2 + \cdots + |c_n|^2 = \|v\|^2,$$

which of course implies (since the norm is a nonnegative real number) that $\|Tv\| \leq \|v\|$.

4. For each part, we use the following facts: T is a projection iff $\mathbb{C}^3$ decomposes as the direct sum of the eigenspaces E_0 and E_1 (recall that E_0 is the null space, and E_1 the range), and it's an orthogonal projection if furthermore $E_0 \perp E_1$. This gives the following answers: a. Orthogonal Projection; b. Orthogonal Projection; c. Projection (not orthogonal); d. Orthogonal Projection.

5. First we have to find some eigenvalues and eigenvectors. By either inspection or direct calculation we find that the vectors $e_1 - e_2, e_2 - e_3, e_3 - e_4$, and $e_4 - e_5$ are eigenvectors with eigenvalue 0, and $e_1 + e_2 + e_3 + e_4 + e_5$ is an eigenvector with eigenvalue 10. But now we have a basis of eigenvectors, which means that the Jordan normal form of T is diagonal, with diagonal entries $0, 0, 0, 0, 10$. From this Jordan form we find the characteristic polynomial is $z^4(z - 10)$, and the minimal polynomial is $z(z - 10)$.

6. T_1 is symmetric, hence diagonalisable. Therefore, it can't have the desired minimal polynomial. T_2 is nilpotent, hence has exactly one eigenvalue (=0), so it can't have ± 1 as eigenvalues. T_3 has the desired minimal polynomial. T_4 has the desired minimal polynomial. T_5 is symmetric, hence diagonalisable. T_6 has the desired minimal polynomial.

7. $\dim range(T - 1) = 6$ implies that $\dim null(T - 1) = 8 - 6 = 2$, so there are two 1-Jordan blocks. As $null(T - 1)^2 \cap null(T - 2)^3 = \{0\}$, and the dimensions add up to 8, we see that

$$\mathbb{C}^8 = null(T - 1)^2 \oplus null(T - 2)^3.$$

Hence, the only eigenvalues are $1, 2$. $\dim null(T - 1)^2 > \dim null(T - 1)$ gives that the 1-generalised eigenspace is $null(T - 1)^2$. Hence, the largest 1-Jordan block has size 2×2, and there must be two of them. Since $null(T-2)^3$ is the 2-generalised eigenspace, we have the largest 2-Jordan block has size at most 3. Hence, we have the following possibilities - where $J(\lambda, i)$ denotes an $i \times i$ λ-Jordan block -

$$\begin{bmatrix} J(1,2) & & & \\ & J(1,2) & & \\ & & J(2,3) & \\ & & & J(2,1) \end{bmatrix},$$

$$\begin{bmatrix} J(1,2) & & & \\ & J(1,2) & & \\ & & J(2,2) & \\ & & & J(2,2) \end{bmatrix},$$

$$\begin{bmatrix} J(1,2) & & & & \\ & J(1,2) & & & \\ & & J(2,2) & & \\ & & & J(2,1) & \\ & & & & J(2,1) \end{bmatrix},$$

$$\begin{bmatrix} J(1,2) & & & & & \\ & J(1,2) & & & & \\ & & J(2,1) & & & \\ & & & J(2,1) & & \\ & & & & J(2,1) & \\ & & & & & J(2,1) \end{bmatrix}$$

8. We have $T^2 = 0$ so that the only eigenvalue is 0. Since $null(T) = span(e_2, e_1 - e_3)$, so the dimension is two, we have that there are two Jordan blocks, so that Jordan form is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A Jordan basis is (v_1, v_2, v_3), where we require that

$$T(v_1) = 0,\ T(v_2) = v_1,\ T(v_3) = 0.$$

Thus, we need $v_2 \notin null(T)$. Take $v_2 = e_3$. Then, $v_1 = T(v_2) = e_2$. Finally, we need $v_3 \in null(T)$ so that (v_1, v_2, v_3) is linearly independent. Take $v_3 = e_1 - e_3$. Then, (v_1, v_2, v_3) is a Jordan basis.

9. a. We have

$$U = span(e_1 - e_2, e_2 - e_3).$$

Apply Gram-Schmidt to the basis $(e_1 - e_2, e_2 - e_3)$ to obtain an orthonormal basis (v_1, v_2) of U, where

$$v_1 = \frac{1}{\sqrt{2}}(e_1 - e_2),$$

$$v_2 = \frac{1}{\sqrt{6}}(e_1 + e_2 - 2e_3).$$

b. It is the vector

$$u = ((e_1 + e_2) \cdot v_1)\, v_1 + ((e_2 + e_3) \cdot v_2)\, v_2 = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

10. This can be proved: let $w \in range(T)$ be nonzero. Then, (w) is a basis of $range(T)$; extend to a basis $C = (w, w_1, \dots, w_k)$ of W. Let $(v_2, \dots, v_n)$ be a basis of $null(T)$, and extend to a basis $B = (v_1, \dots, v_n)$ of V (we know that $\dim null(T) = \dim V - \dim range(T) = \dim V - 1$). Then, we have the matrix of T relative to B and C is

$$[T]_B^C = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Consider the linear functional $f : V \to \mathbb{C}$ defined on the basis B as

$$f(v_2) = \cdots = f(v_n) = 0 \in \mathbb{C}, \text{ and } f(v_1) = a_1.$$

This defines a linear functional on V and, if $v = \sum_{j=1}^n b_j v_j \in V$, then

$$T(v) = b_1 a_1 w = f(v) w.$$